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Analytical Study on Free Vibration Dynamics of Compressible Fluids in Rigid-Walled Rectangular Tanks

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Abstract

This paper calculates the natural frequencies using analytical expressions derived from the Helmholtz equation, with boundary conditions adapted for rigid-walled rectangular tanks. These analytical methods facilitate the examination of how variations in tank dimensions (length, height, and width) influence the frequencies via sensitivity analysis. Results indicate that increases in the length, height, or width of the tank consistently result in a decrease in natural frequencies across all modes. This foundational study advances the understanding of fluid-structure interaction (FSI) in engineering applications such as storage tanks and seismic-resistant structures and provides valuable insights for future research.

Keywords: Free vibration, Exact solution, Fluids.

1. Introduction

Understanding how fluids and structures interact (fluid-structure interaction, FSI) is crucial in engineering applications such as storage tanks, pipelines, and offshore platforms. This paper investigates the free vibration of compressible fluids in rigid rectangular tanks, emphasizing the importance of FSI.

The fluid pressure on structures like dams and storage tanks plays a critical role in earthquake resistance. Numerous studies have explored the dynamic behavior of these liquid-filled structures during seismic events, considering their significance and the complex interplay between soil, structure, and fluid. Researchers have consistently explored how liquid storage tanks respond to seismic forces. This understanding not only enhances knowledge of their post-earthquake performance but also provides broader insights.

The journey began in 1933 with Westergaard's solution for hydrodynamic pressure on rigid dam walls under harmonic excitation. Later, Housner and Jacobsen (1934) reported the first experimental and analytical findings on rigid tanks subjected to horizontal earthquake motion. Lemm (1945) tackled the classic sloshing wave problem in rigid cylindrical tanks. Since then, researchers like Graham and Rodriguez (1964), Housner (1957, 1963), and many others (Bauer, 1967; Hunt & Priestley, 1971; Fischer, 1974; Haroun & Housner, 1985, 1986; Loft, 1985; Chen et al., 1995; Chen & Kianoush, 1996, 1997; Bayraktar et al., 2006) have further explored how liquid-filled tanks react to earthquakes, considering factors such as tank shape, fluid properties, wall flexibility, and sloshing dynamics.

The field has been significantly advanced by analytical and numerical modeling of FSI in these tanks using techniques like Rayleigh-Ritz and finite element methods. Key contributions include Zou's work (Zou, 1997, 2002) on analytical modeling, Amabili's work (Amabili & Di Carlo, 2001, 2004) on numerical methods, and Raouf Ibrahim's comprehensive compilation (Ibrahim, 2005) of sloshing wave breakthroughs. Recent contributions (Idriss & Baidillah, 2000; Carpinteri et al., 2001; Jamalabadi, 2019; Kioka et al., 2013) continue to deepen our understanding.

In 2023, Ismail introduced a simple 1-DOF model for FSI analysis, offering a valuable tool for exploring basic concepts (Ismail, 2023). Łojek et al.'s (2022) study highlights the importance of including FSI effects in analyzing fluid behavior. FSI research progress has also led to advanced numerical methods and models. Meng et al. (2022) presented a hydroelastic FSI solver, while Xie et al. (2023) explored a novel bearing's lubrication performance using both experiments and FSI analysis. Han et al. (2022) worked on reduced-order models with deep neural networks showcases the growing role of machine learning in enhancing FSI simulation accuracy.

More recently, advancements in computational modeling, such as finite element methods (FEM) and computational fluid dynamics (CFD), have enhanced our ability to simulate and analyze FSI in complex systems. For instance, Hariri Asli et al. (2023) applied CFD to study the influence of partition walls on hydraulic parameters in reservoirs, demonstrating how modifications like baffles can optimize fluid retention times and improve structural resilience (Hariri et al., 2023). Such studies underscore the need to account for both fluid dynamics and structural considerations when designing resilient systems.

Our study employs an analytical approach to solve the governing equations for 2D and 3D rectangular tanks, leading to clear expressions for natural frequencies of the velocity potential. These expressions provide a solid foundation for understanding FSI behavior of compressible fluids in rectangular tanks. We further incorporate the Rayleigh-Ritz method to account for fluid compressibility effects. This combined approach offers a comprehensive framework for analyzing the free vibration of compressible fluids in rectangular tanks, considering FSI effects.

The paper is structured as follows: Section 2 presents the governing equations and boundary conditions, clarifying the use of velocity potential in the Helmholtz equation. Section 3 details the solution method, including the transformation of the general solution into eigenvalue results. Section 4 presents the results of our sensitivity analyses, discussing the impact of fluid properties and tank dimensions on resonance frequencies.

Finally, Section 5 concludes the study and suggests directions for future research.

2. Governing Equations and Boundary Conditions

In this section, we present the mathematical framework governing the behavior of compressible fluids within rectangular tanks. These equations are fundamental to understanding the dynamics of fluid-structure interaction (FSI) phenomena.
The core principle governing fluid motion is the conservation of mass. For a compressible, inviscid fluid, this principle is expressed mathematically using the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{1}$$

where ρ is the fluid density, t is time, and $\mathbf{v}$ is the velocity vector.
For irrotational flow, we can express the velocity vector in terms of a scalar velocity potential ϕ:

$$v = \nabla \tag{2}$$

Substituting this into the mass conservation principle and rearranging, we derive:

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 0 \tag{3}$$

where c is the speed of sound in the fluid.
Assuming harmonic time dependence of the form $e^{i\omega t}$, where ω is the angular frequency, we arrive at the Helmholtz equation for the velocity potential:

$$\nabla^2 \phi + k^2 \phi = 0 \tag{4}$$

where $k = \omega/c$ is the wave number.

2.1. Three-Dimensional Tank

For a three-dimensional rectangular tank with length L, height H, and width B, the Helmholtz equation becomes:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} + k^2 \phi = 0 \tag{5}$$

The boundary conditions for the three-dimensional tank are:

1. No flow through the rigid walls:
$\frac{\partial \phi}{\partial n} = 0$ on all boundaries, where n is the outward unit normal vector.

2. Continuity of velocity potential at the fluid-structure interface:
$\phi = 0$ at the free surface (if applicable).

3. Pressure-displacement relationship at the fluid-structure interface:
$\rho \frac{\partial \phi}{\partial t} = -p$ at the interface, where p is the pressure

These conditions specify the behavior of the velocity potential at the tank boundaries, ensuring no flow through the walls and appropriate coupling between the fluid and structure.

2.2. Analysis of Fluid-Structure Interaction

To solve the Helmholtz equation subject to the given boundary conditions, we employ the method of separation of variables. We assume a solution of the form:

$$\phi(x, y, z) = X(x)Y(y)Z(z) \tag{6}$$

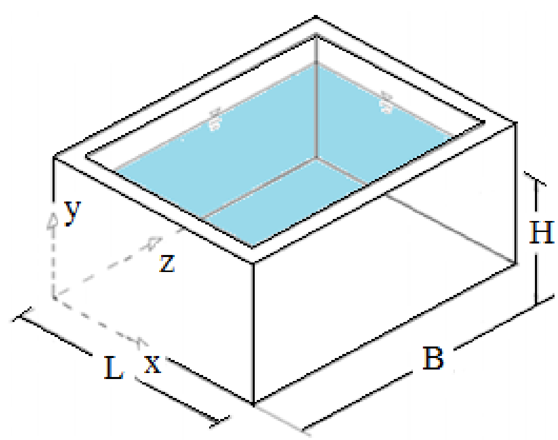

Fig. 1 Solid Three-Dimensional Rectangular Cuboid Reservoir Containing Fluid

Substituting this into the Helmholtz equation and applying the boundary conditions leads to the following general solution for the velocity potential:

$$\phi(x,y,z)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}A_{mnp}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{H}\right)\cos\left(\frac{p\pi z}{B}\right) \quad (7)$$

where A_{mnp} are amplitude coefficients, and m, n, and p are non-negative integers.

The eigenvalue equation relating the wave number to the tank dimensions is:

$$k^2=\left(\frac{m\pi}{L}\right)^2+\left(\frac{n\pi}{H}\right)^2+\left(\frac{p\pi}{B}\right)^2 \quad (8)$$

From this, we can derive the natural frequencies of the system:

$$f_{mnp}=\frac{c}{2\pi}\sqrt{\left(\frac{m\pi}{L}\right)^2+\left(\frac{n\pi}{H}\right)^2+\left(\frac{p\pi}{B}\right)^2} \quad (9)$$

This equation clearly shows the dependence of the natural frequencies on both the fluid properties (through the speed of sound c) and the tank dimensions (L, H, and B) (see Fig. 1).

In the next section, we will present a sensitivity analysis to examine how changes in tank dimensions affect these natural frequencies.

3. Sensitivity Analysis for Three-Dimensional Fluid

Using the equations developed in Section 2, we calculate the natural frequencies for various tank dimensions. The equations, derived from fluid mechanics and adapted for our 3D tank model, allow us to analyze how the tank's size (length, height, and width) influences its vibration behavior. By examining various tank geometries, we can investigate how a tank's dimensions affect its natural frequencies.

Before presenting the results, it's important to note the following:

1. The natural frequencies are calculated for the velocity potential, not the velocity field itself.

2. The fluid properties, specifically the speed of sound c, directly impact the natural frequencies as shown in Equation (9).

3. The compressibility of the fluid is accounted for through the wave equation and the resulting Helmholtz equation.

4. While the boundary conditions assume no flow through the walls, this does not imply incompressibility near the walls. The fluid remains compressible throughout the domain.

5. The separation of variables method used does not imply that bulk waves do not interact. Rather, it provides a mathematical framework for solving the Helmholtz equation.

The following sections present sensitivity analyses. Each section explores how changing one dimension (length in Section 3.1, height in Section 3.2, and width in Section 3.3) affects the tank's first five natural frequencies, while keeping the other two dimensions constant.

3.1. Effect of Length Variation on First Five Natural Frequencies

Table 1 showcases the effect of varying the tank length on the first five natural frequencies, with height and width held constant at 3 units each.

Figure 2 illustrates this trend graphically. As the length increases, we observe a consistent decrease in natural frequencies across all modes except the first. This behavior aligns with our theoretical expectations: a larger tank dimension allows for longer wavelengths, corresponding to lower frequencies.

The first mode (Mode 1) remains constant because it corresponds to the fundamental frequency in the height and width directions, which are kept constant in this analysis.

3.2. Effect of Height Variation on First Five Natural Frequencies

Table 2 presents the effect of varying the tank height on the first five natural frequencies, with length and width held constant at 4 and 3 units respectively.

Table 1 Effect of Length Variation on the First Five Natural Frequencies of Three-Dimensional Fluid (H=3, B=3)

Length (L)	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
2	268.33	432.67	449.00	562.85	646.22
3	268.33	360.00	432.67	494.77	549.91
4	268.33	323.11	432.67	449.00	468.61
5	268.33	304.53	393.63	432.67	456.00
6	268.33	293.94	360.00	432.67	449.00
7	268.33	287.37	338.11	408.92	432.67
8	268.33	283.02	323.11	380.66	432.67
9	268.33	280.00	312.41	360.00	417.61
10	268.33	277.82	304.53	344.46	393.63

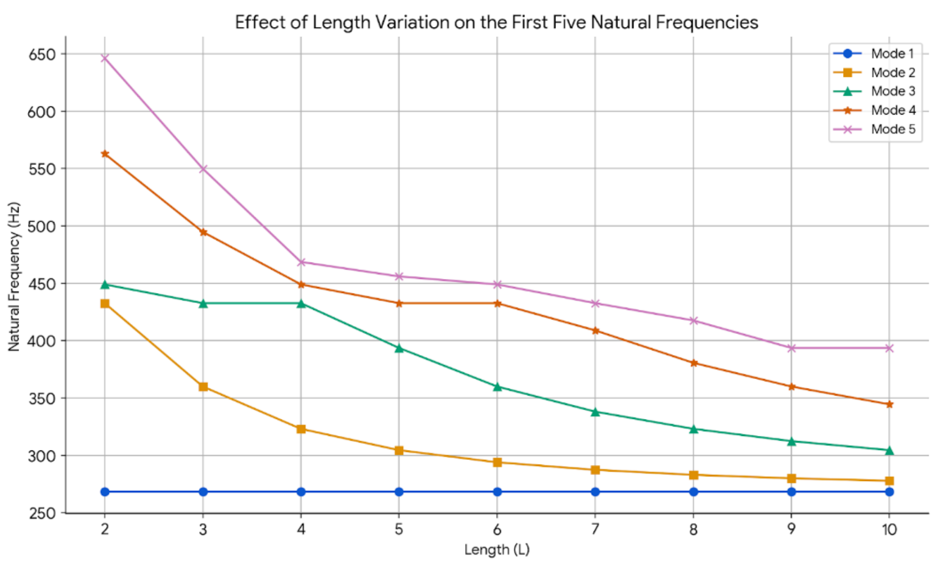

Fig. 2 Effect of Length Variation on the First Five Natural Frequencies.

Table 2 Effect of Height Variation on the First Five Natural Frequencies of Three-Dimensional Fluid (L=4, B=3)

Height (H)	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
2	300.00	349.86	468.61	590.93	617.74
3	268.33	323.11	432.67	449.00	468.61
4	256.32	313.21	361.25	403.61	441.93
5	250.57	308.52	322.89	369.67	432.67
6	247.39	300.00	305.94	349.86	384.19
7	245.45	285.31	304.38	337.35	351.74
8	244.18	275.36	303.36	328.98	328.98
9	243.31	268.32	302.65	312.41	323.11
10	242.68	263.18	300.00	302.15	318.85

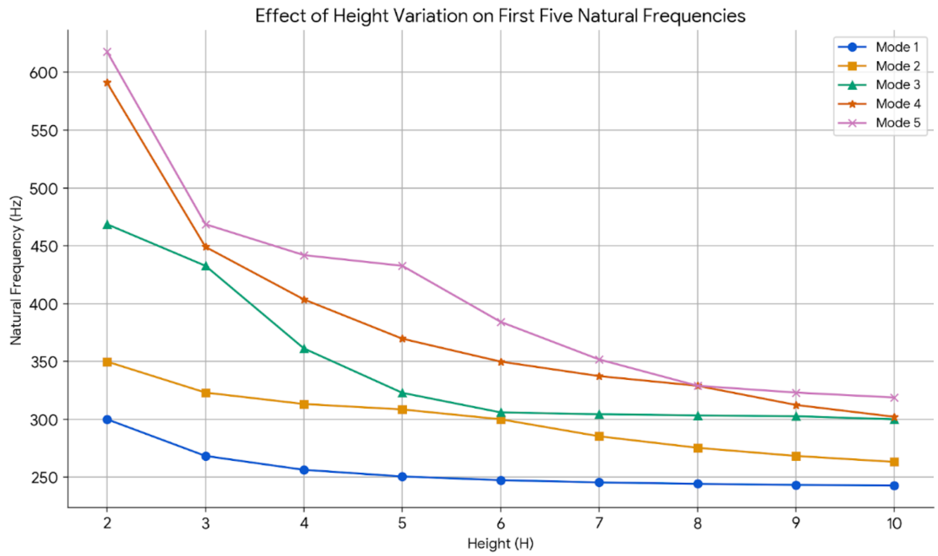

Fig. 3 Effect of Height Variation on the First Five Natural Frequencies

Figure 3 illustrates these results graphically. Similar to the length variation, increasing the height leads to a decrease in natural frequencies across all modes. This trend is consistent with our theoretical understanding: increasing any dimension allows for longer wavelengths and thus lower frequencies.

The rate of decrease is most pronounced for the higher modes, which are more sensitive to changes in tank dimensions.

3.3. Effect of Width Variation on First Five Natural Frequencies

Table 3 presents the effect of varying the tank width on the first five natural frequencies, with length and height held constant at 4 and 3 units respectively.

Figure 4 illustrates these results graphically. Consistent with the trends observed for length and height variations, increasing the width leads to a decrease in natural frequencies across all modes.

The effect is most pronounced for the lower modes, particularly Mode 1, which shows the steepest decline as width increases. This is because the fundamental frequency in the width direction is most directly affected by changes in this dimension.

Table 3 Effect of Width Variation on the First Five Natural Frequencies of Three-Dimensional Fluid (L=4, H=3, B=Variable)

Width (B)	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
2	379.47	420.00	509.12	523.07	540.00
3	268.33	323.11	432.67	449.00	468.61
4	216.33	281.42	402.49	420.00	440.91
5	187.45	259.88	387.73	405.88	427.48
6	169.71	247.39	379.47	379.99	420.00
7	158.05	239.54	374.41	393.17	415.43
8	150.00	243.31	371.08	390.00	412.43
9	144.22	230.65	368.78	387.81	410.37
10	139.94	228.00	376.13	386.24	408.88

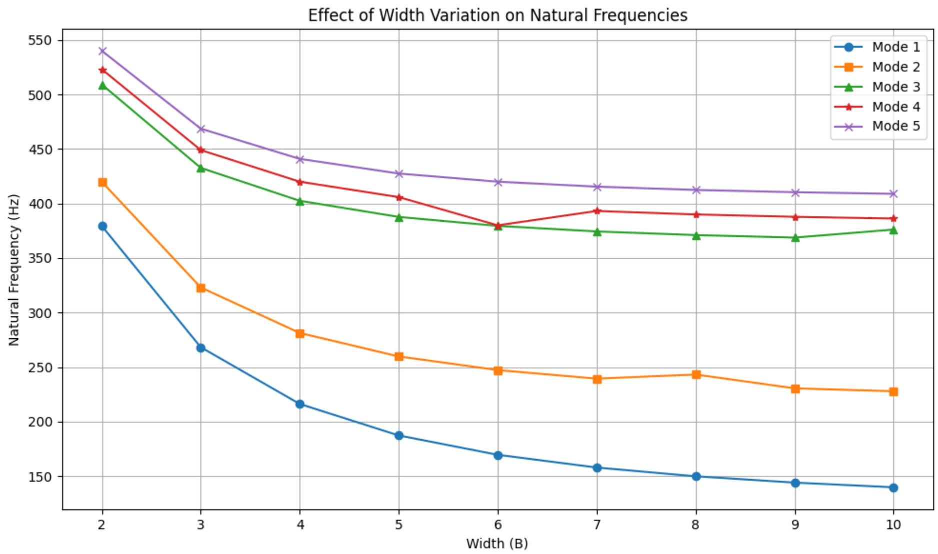

Fig. 4 Effect of Width Variation on the First Five Natural Frequencies

3.4. Impact of Fluid Properties

It's important to note that while our tables and figures show the relative changes in natural frequencies due to tank dimension variations, the absolute values of these frequencies also depend on the speed of sound in the fluid, c, as shown in Equation (9):

$$f_{mnp} = \frac{c}{2\pi}\sqrt{\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{H}\right)^2 + \left(\frac{p\pi}{B}\right)^2} \tag{10}$$

The speed of sound c is a function of the fluid's compressibility and density. For an ideal gas, it can be expressed as:

$$c = \sqrt{\frac{\gamma RT}{M}} \tag{11}$$

where γ is the ratio of specific heats, R is the universal gas constant, T is the absolute temperature, and M is the molar mass of the gas.

For liquids, the speed of sound is typically given by:

$$c = \sqrt{\frac{K}{\rho}} \tag{12}$$

where K is the bulk modulus of elasticity and ρ is the density of the liquid.

These relationships demonstrate how fluid properties directly influence the natural frequencies of the system. Changes in temperature, pressure, or fluid composition that affect the speed of sound will proportionally scale all natural frequencies.

4. Results and Discussion

Our investigation into the world of three-dimensional fluid reservoirs has yielded fascinating results regarding their natural frequencies. The data in this section reveals a captivating trend: as the dimensions of the reservoir increase, its natural frequencies tend to decrease.

This trend holds true across variations in length, height, and width of the reservoir, as illustrated in Figures 1, 2, and 3. Key observations include:

1. Increasing any dimension of the reservoir leads to lower natural frequencies across most modes.

2. The first mode (Mode 1) remains constant when varying length, as it corresponds to the fundamental frequency in the height and width directions.

3. The effect of dimensional changes is more pronounced for higher modes in length and height

variations, but for lower modes in width variations.

4. The relationship between tank dimensions and natural frequencies is non-linear, with the rate of frequency decrease diminishing as dimensions increase.

These observations can be explained by considering the underlying physics:
1. Larger dimensions allow for longer wavelengths, which correspond to lower frequencies.

2. The compressibility of the fluid plays a crucial role. When the reservoir dimensions increase, the fluid has a larger volume for deformation, reducing the overall stiffness of the system.

3. The boundary conditions, which specify no flow through the walls, interact with the compressibility of the fluid to produce the observed mode shapes and frequencies.

4. While we used the separation of variables method to solve the Helmholtz equation, this does not imply that bulk waves do not interact in reality. The method provides a mathematical framework for understanding the system's behavior.

The dependence of natural frequencies on fluid properties, particularly the speed of sound, highlights the importance of considering fluid characteristics in tank design. Changes in temperature, pressure, or fluid composition can significantly alter the resonant behavior of the system.
These findings have important implications for the design and analysis of fluid-containing structures, particularly in seismic engineering applications. Understanding how tank dimensions and fluid properties influence natural frequencies can help engineers optimize designs to avoid resonance with expected excitation frequencies, improving the safety and performance of these structures.

5. Conclusions and Future Work

This study presents a comprehensive analytical approach to understanding the free vibration dynamics of compressible fluids within rigid-walled rectangular tanks. By deriving and solving the governing equations for two-dimensional and three-dimensional tank geometries, we have gained valuable insights into the behavior of these systems.

Key findings of our study include:

1. The natural frequencies of the fluid in a rectangular tank depends on both the tank dimensions and the fluid properties, specifically the speed of sound in the fluid.

2. Increasing any tank dimension (length, height, or width) generally results in a decrease in natural frequencies across most modes. This occurs due to the allowance for longer wavelengths in larger tanks.

3. The effect of dimensional changes on natural frequencies is non-linear, with the rate of frequency decrease diminishing as dimensions increase.

4. The compressibility of the fluid significantly influences the natural frequencies and mode shapes of the system.5. Boundary conditions specifying no flow through the rigid walls interact with fluid compressibility to produce the observed frequency patterns. These findings have significant implications for the design and analysis of fluid-containing structures, particularly in seismic engineering applications. By understanding how tank dimensions and fluid properties influence natural frequencies, engineers can optimize designs to avoid resonance with expected excitation frequencies, thereby improving the safety and performance of these structures.

However, it is important to note the limitations of this study:

1. The study assumes rigid tank walls; however, real-world tanks may exhibit some flexibility, which could affect the system's dynamic behavior.

2. The analysis focuses solely on rectangular tanks, while many real-world tanks have different geometries, such as cylindrical or spherical shapes.

3. Only free vibrations are considered, whereas forced vibrations, such as those induced by seismic activity, may introduce additional complexities.

4. The effects of gravity and surface waves, which could play a significant role in certain applications, are not included. Given these limitations and the importance of this field, we propose the following directions for future research:

1. Extend the analysis to include flexible tank walls, incorporating more comprehensive fluid-structure interaction effects.

2. Investigate different tank geometries, such as cylindrical and spherical tanks, to broaden the applicability of the findings.

3. Analyze forced vibrations, particularly those induced by seismic activity, to better understand the system's response to real-world excitations.

4. Include gravity effects and surface wave phenomena to model the behavior of liquids in partially filled tanks more accurately.

5. Develop numerical models to complement and extend the analytical approach, enabling the study of more complex geometries and boundary conditions.

6. Experimentally validate the analytical and numerical results to ensure their relevance to real-world scenarios.

7. Explore the effects of internal baffles or other structural modifications on the natural frequencies and mode shapes of the fluid.

8. Study the impact of fluid stratification or multi-phase fluids on the system's dynamic behavior.

9. Analyze energy dissipation mechanisms in the fluid and their influence on the system's damping characteristics.

10. Investigate active and passive control strategies to mitigate unwanted vibrations in fluid-filled tanks. This study establishes a solid foundation for understanding the free vibration dynamics of compressible fluids in rigid-walled rectangular tanks. By addressing the identified limitations and pursuing the proposed research directions, future work can further enhance our understanding of these complex systems and contribute to the development of safer and more efficient fluid-containing structures.

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